Notes on Examples of SU(2),SO(3)-actions on Positively Curved 6-manifolds

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In this article we want to list a few examples of isometric $SU(2), SO(3)$-actions on all known examples of positively curved 6-manifolds, namely S^6, $\mathbb{CP}^3$, $SU(3)/T^2$, $SU(3)//T^2$. We list the isotropy groups and give geometric description of the orbit spaces.

1. $SO(3)$-action on S^6, given by $A(\vec{x}, \vec{y}, z) = (A\vec{x}, A\vec{y}, z),\ A \in SO(3),\ \vec{x}, \vec{y} \in \mathbb{R}^3,\ z \in \mathbb{R},\ (\vec{x}, \vec{y}, z) \in S^6$. This action has 2 isolated fixed points $(0, 0, \pm 1)$. The orbit space is a 3-ball, where the interior points correspond to principal orbits with trivial isotropy, the boundary 2-sphere minus 2 points corresponds to singular orbits with $SO(2)$ isotropy, and 2 antipodal points correspond to 2 fixed points. The singular orbits consist of points whose $\vec{x}$, $\vec{y}$-components are linearly dependent but not both zero. Since the orbit space has boundary, we may apply the Soul Theorem.

2. $SO(3)$-action on $\mathbb{CP}^3$, given by $A(z_1 : z_2 : z_3 : z_4) = (A(z_1 : z_2 : z_3)^T : z_4),\ A \in SO(3),\ (z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^3$. This action has a unique isolated fixed point $(0 : 0 : 0 : 1)$. All the isotropy types (and the corresponding points) are:

1. $SO(3)$: isolated fixed point $(0 : 0 : 0 : 1)$;
2. $SO(2)$: $(\vec{x} \in \mathbb{R}^3 : w \neq 0 \in \mathbb{C})$;
3. $O(2)$: $(\vec{x} \in \mathbb{R}^3 : 0)$, this is a unique orbit;
4. $SO(2)$: $(\vec{v} : 0)$, where $\vec{v} \in \mathbb{CP}^2$ satisfies $(v, v) = \sum_{i=1}^{3} v_i^2 = 0$, this is also a unique orbit;
5. $\mathbb{Z}/2$: $(\vec{v} : 0)$, where $\vec{v} \in \mathbb{CP}^2 \setminus \mathbb{RP}^2$ satisfies $(v, v) = \sum_{i=1}^{3} v_i^2 \neq 0$;
6. id: $(\vec{v} \in \mathbb{C}^3 \setminus \mathbb{R}^3 : w \neq 0 \in \mathbb{C})$.

The orbit space in this case is a 3-ball, where interior points minus a line segment correspond to principal orbits id, the boundary 2-sphere minus 2 points correspond to $SO(2)$-singular orbits (type 2), 2 points on the boundary correspond to $SO(3)$-orbit (fixed point) and $O(2)$-orbit respectively, and a line segment in the interior corresponds to $\mathbb{Z}/2$-orbits connecting $O(2)$-orbit and $SO(2)$-orbit (type 4), and lastly a unique point in the interior corresponds to $SO(2)$-orbit (type 4).

For curiosity I applied Soul Theorem to the orbit space under Fubini-Study metric on $\mathbb{CP}^3$. According to my computation, the soul orbit, which is the orbit of maximal distance to the boundary 2-sphere, must consist of points whose last coordinate is 0, thus must be of type 3,4, or 5. I suspect it is of type 4, i.e. the unique $SO(2)$-orbit in the interior, but I didn't finish my computation. In order to compute the distance from an interior point to the boundary, I used Lagrange multiplier, but the equations seem very complicated.

One question: from q-extent argument, we know that an $SO(3)$-action on positvely curved 6-manifolds can have at most 3 isolated fixed points, if it has one. Now we have examples with 1 and 2, how about 3?

3. Another $SO(3)$-action on $\mathbb{CP}^3$, induced from one $SU(2)$-action. Let $A \in SU(2)$ act on $\mathbb{CP}^3$ via $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y}),\ \vec{x}, \vec{y} \in \mathbb{C}^2$. This action is ineffective since $-Id \in SU(2)$ acts trivially, thus descends to an $SO(3)$-action. The orbit space is a 3-ball, where interior points correspond to principal orbits id, and the boundary points correspond to singular $U(1)$-orbits. The singular orbits consist of points whose $\vec{x},\ \vec{y}$-components are linearly dependent.

4. Another $SO(3)$-action on $\mathbb{CP}^3$, coming from the 4-dim complex irrep of $SU(2)$. The irreducible action of $SU(2)$ on $\mathbb{C}^4$ induces one on $\mathbb{CP}^3$, which is ineffective with kernel $\mathbb{Z}/2$ and descends to $SO(3)$. We realize $\mathbb{C}^4 = span_{\mathbb{C}}\{x^3, x^2y, xy^2, y^3\}$ and write $(a, b, c, d) = ax^3 + bx^2y + cxy^2 + dy^3$. Under this identification, $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2),\ |a|^2 + |b|^2 = 1$ corresponds to

$$\begin{bmatrix} a^3 & -a^2\bar{b} & a\bar{b}^2 & -\bar{b}^3 \\ 3a^2b & a(3|a|^2-2) & \bar{b}(1-3|a|^2) & 3\bar{a}\bar{b}^2 \\ 3ab^2 & b(3|a|^2-1) & \bar{a}(3|a|^2-2) & -3\bar{a}^2\bar{b} \\ b^3 & \bar{a}b^2 & \bar{a}^2b & \bar{a}^3 \end{bmatrix}. \tag{1}$$

The orbit space is a 3-sphere, and the orbit types are as follows:

1. the principal isotropy is trivial.
2. 2 singular orbits with $U(1)$-isotropy, corresponding to $x^3,\ x^2y$ respectively. The slice representation of $U(1)$ at the singular orbit x^3 has slope $(2, 3)$ and the slice rep at x^2y has slope $(1, 2)$, which matches up with exceptional orbits.
3. an exceptional orbit with isotropy S_3, with representative $(1, 0, 0, 1) \sim (1, 0, 3, 0)$. This S_3 is the image of the binary dihedral group of order 12 generated by $\begin{bmatrix} \zeta_6 & 0 \\ 0 & \bar{\zeta}_6 \end{bmatrix}, \begin{bmatrix} 0 & \zeta_{12} \\ -\bar{\zeta}_{12} & 0 \end{bmatrix} \in SU(2)$. The slice is the real linear span of $(1, 0, 0, -1), (0, 1, 1, 0), (0, i, -i, 0)$, where S_3 acts on $(1, 0, 0, -1)$ as reflection and acts as the dihedral group on the span of the other 2 vectors. One axis of reflection in the dihedral group is $(0, (1 - \sqrt{3}i), (1 + \sqrt{3}i), 0)$.
4. a 1-parameter families of exceptional orbits with isotropy $\mathbb{Z}/2$ with representatives $(1, 0, t, 0)$ where $t \in \mathbb{R}_{>0},\ t \neq 3$, whose orbit stratum consists of a path connecting the singular x^3 to the exceptional orbit 3 and another path connecting 3 to x^2y. Note: $(1, 0, t, 0)$ lies in the same orbit as $(0, 1, 0, t)$, and $(0, 1, 1, 0)$ lies in the orbit of $(1, 0, 1, 0)$.
5. a 1-parameter families of exceptional orbits with isotropy $\mathbb{Z}/3$ with representatives $(1, 0, 0, \mu) \sim (1, 0, 0, \frac{1}{\mu}),\ 0 < |\mu| < 1$. The orbit stratum is a path connecting the singular orbit x^3 and the exceptional orbit 3 $(1, 0, 0, 1)$.

Remark 1 *I computed the eigenvalues and eigenvectors of the Lie algebra action to determine singular orbits, and by diagonalizing $SU(2)$ matrices we know that exceptional isotropy near singular orbits must consist of matrices with "special" eigenvalues, i.e. 4-th or 6-th root of unity.*

5. $SU(2)$-action on S^6 given by $A(\vec{x},\vec{y}) = (A\vec{x}, A\vec{y}),\ A \in SU(2),\ \vec{x} \in \mathbb{R}^4,\ \vec{y} \in \mathbb{R}^3,\ (\vec{x},\vec{y}) \in S^6$. The action on the $\vec{x}$-component comes from the real 4-dim irrep of $SU(2)$, i.e. the realification of the standard $SU(2)$-action on $\mathbb{C}^2$, and the action on $\vec{y}$-component comes from the standard $SO(3)$-action on $\mathbb{R}^3$. The orbit space is the 3-sphere, with only one singular orbit corresponding to the points whose $\vec{x}$-component vanishes, and the singular isotropy is $U(1)$. The principal isotropy is trivial.

Remark 2 *In this case, if we are given the orbit structure, then by van Kamper theorem we know the G-space is simply connected, and by Mayer-Vietoris sequence we know it's a homology sphere. Thus by the solution to Poincare conjetcure we know the G-space must be homeomorphic to the 6-sphere.*

6. $SU(2)$-action on the homogeneous flag manifold $SU(3)/T^2$, given by left multiplication. The orbit space is a 3-sphere with 3 singular orbits with $U(1)$-isotropy corresponding to the matrices

$$Id,\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The principal isotropy is trivial.

7. $SU(2)$-action on the biquotient $SU(3)//T^2 = (z,w,zw) \setminus SU(3)/(1,1,z^2w^2)^{-1},\ z,w \in S^1$. $SU(2)$ acts from the right as the first 2 block of $SU(3)$, commuting with T^2-action. The orbit space is again a 3-sphere, with trivial principal isotropy and 3 singular $U(1)$-orbits with representatives

$$[Id] = \begin{bmatrix} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} A & 0 \\ 0 & \bar{z}\bar{w} \end{bmatrix},\ \begin{bmatrix} 0 & 0 & \bar{z}\bar{w}^2 \\ wa_{21} & wa_{22} & 0 \\ zwa_{11} & zwa_{12} & 0 \end{bmatrix},\ \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix}.$$

And the interesting thing is that in this case we have a 1-parameter family of exceptional orbits with isotropy $\mathbb{Z}/3$, connecting the second and the third singular orbits above. The representatives of exceptional orbits are

$$\begin{bmatrix} c_1a_1 & c_1a_2 & f_1 \\ c_2a_1 & c_2a_2 & f_2 \\ b_1 & b_2 & 0 \end{bmatrix} \in SU(3). \tag{2}$$

Note: when c_1(resp. c_2) becomes 0, we get back the second (resp. the third) singular orbit. The vectors $(a_1, a_2),\ (b_1, b_2)$ are actually (left) eigenvectors of elements in the exceptional isotropy.

Remark 3 *I computed the $U(1)$-fixed point set in this and the previous example. It is a set of 6 isolated points, namely 2 antipodal points from each singular orbit (homeomorphic to S^2). This matches up with the conclusion that the Euler characteristic of fixed point set under torus action is the same as that of the whole manifold.*

Remark 4 *If we are given the orbit structure, by Mayer-Vietoris sequence the G-space has the same Betti numbers as $SU(3)//T^2$, i.e. $b_0 = b_6 = 1,\ b_2 = b_4 = 2, b_{2i+1} = 0$.*

8. $SO(3)$-action on $SU(3)/T^2$, given by left multiplication. This action has orbit space a 3-sphere, with trivial principal isotropy. The principal orbits consist of matrices all of whose 3 columns are not proportional to real vectors. For simplicity, we say they are "complex" vectors and if a vector is proportional to a real vector, we just call it real vector. There are 3 singular orbits with $SO(2)$-isotropy, and one of them has representatives $[\vec{v}, \bar{\vec{v}}, \vec{w}] \in SU(3)$, where $\vec{v} \in \mathbb{C}^3$, $\vec{w} \in \mathbb{R}^3$ are column vectors, $\bar{\vec{v}}$ is the complex conjugation of $\vec{v}$, and $\vec{v}$ satisfies $(\vec{v}, \vec{v}) = \sum_{i=1}^3 v_i^2 = 0$. The other two have similar representatives with $\vec{w}$ put in the first and second column.
Exceptional orbits are also interesting in this example. There is one exceptional orbit with $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-isotropy, corresponding to the $SO(3)$ matrices in $SU(3)$. Besides that, there are three 1-parameter families of $\mathbb{Z}/2$-orbits. Their representatives are like $[\vec{v_1}, \vec{v_2}, \vec{w} \in \mathbb{R}^3] \in SU(3)$, where $(\vec{v_1}, \vec{v_1}) \neq 0$ ($(\cdot, \cdot)$ is the complex linear extension of real inner product). Of course the column $\vec{w}$ can be permuted around. In the orbit space, these three families of $\mathbb{Z}/2$-orbits are line segments connecting the three singular orbits with the $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-orbit. The whole picture has a 3-fold symmetry, which might have something to do with the Weyl symmetry.